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On Gains to Bid-Takers from Using Multi-Stage Auctions

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Abstract:

Each year, multi-stage auctions of one form or another sell or let billions of dollars worth of goods and contracts. Yet despite the significance of such auctions, the existing theory of auctions and competitive bidding fails to explain why a bid-taker might prefer a multi-stage auction to a, possibly simpler, single-stage mechanism. In fact, the existing theory tends to overlook multi-stage mechanisms altogether.

This paper takes a first step in correcting that omission. We start with several illustrations of what we will define to be multi-stage auctions. All the illustrated auctions allow bidders to acquire information--or, more generally, to increase the amount of resources committed to bidding--in stages. We conjecture that allowing such a sequential commitment of resources could result in a more efficient auction than if bidders had no chance to adjust their commitments.

An analytic example illustrates and documents the benefits to the bid-taker from using a multi-stage auction. In the example, bidders acquire costly information in stages; a bidder may stop acquiring additional information as soon as further participation in the auction would no longer increase his expected net profit. The resulting auction generates significantly more expected revenue for the bid-taker than if the bidders all had to acquire the same amount of information.

Introduction:

Each year, multi-stage auctions of one form or other sell or let a variety of goods and contracts with a combined value in the billions of dollars. For example, before starting the progressive oral bidding for a rare stamp, the bid-taker may have already accepted written "book bids." In some real estate auctions, the bid-taker allows bidders to raise their bids orally once the sealed bids have been opened; the Government sells timber rights in much the same way. Americana Arts Auctions (Guarino, 1985) accepts mailed bids, but in the final week before the deadline, will also report the current high bid--and allow a bidder to make a bid--over the telephone; the bidding stops once the telephone has been inactive for at least five minutes after the specified deadline. (Here, the mailed bid may be of the form "raise my bid by the minimum increment, as needed, so long as it does not go above x;" this gives the auction a distinct, and unusual second-price flavor.) Cassady (1967) describes several multi-stage schemes in his compendium of auctions and competitive bidding.

If we view written proposals as a generalized form of bids, then the letting of grants and contracts provides additional illustrations of multi-stage auctions. For example, in developing a new defense system, the Government may first let a number of contracts to develop prototypes for the system, and then select a final winner from among the prototype developers. In a similar vein, IBM awarded grants, each worth millions of dollars, through a two-stage process in which academic institutions first submitted abbreviated proposals that determined who would be funded (by IBM) to develop more detailed proposals.

Despite this significant use of multi-stage auctions, the existing theory of auctions and competitive bidding fails to explain why a bid-taker might prefer a multi-stage mechanism over a possibly simpler single stage one. In fact, as we will see, the existing theory does not argue against the use of multi-stage mechanisms; the existing theory focuses on single-stage mechanisms and largely ignores the possibility of multi-stage mechanisms. This paper works toward correcting that omission by examining what actually constitutes a multi-stage auction, suggesting potential benefits, and documenting and illustrating these benefits through an example.

Defining "Multi-Stage":

This section isolates the essence of multi-stage mechanisms. We start by listing various characteristics of the illustrated multi-stage auctions; characteristics shared by all the illustrations may differentiate multi-stage auctions from single-stage mechanisms, while those shared by only a subset of the illustrations may only differentiate one type of multi-stage auction from another. A common characteristic will provide the basis for our definition of multi-stage. The absence of this characteristic in existing models argues that the existing theory ignores multi-stage mechanisms as we will have defined them.

We gave as many illustrations of multi-stage auctions as we did in part to enable us to discern what characteristics differentiate one multi-stage auction from another as opposed to differentiating multi-stage auctions as a group from other auctions. In some cases, the bidder knows the results of earlier stages before proceeding to subsequent stages; in other cases, the bidder only knows that someone might have bid in a previous stage. Thus, we would hope to be able to characterize multi-stage auctions as something more general than simply mechanisms that give a bidder sufficient time to process whatever information others' bids in an earlier stage may reveal about the value of or competition for the object being auctioned. In some cases, one stage determines who will be subsidizied or even allowed to bid in a subsequent round, while in other examples, an individual may bid in a later stage without bidding in earlier stages. In some cases, the type of auction used varies from stage to stage, and in other cases, it doesn't. In some cases, risk aversion may play an important role, but in other cases, the bidders would seem to be relatively risk neutral. In some cases, first round bidders have priority over second round bidders in the case of ties, while in other cases the nature of the bids changes so much from one stage to the next that even the concept of "ties" becomes meaningless. Each of these characteristics differentiate one multi-stage auction from another more than they differentiate multi-stage auctions from other auctions.

Yet these illustrations do all share one characteristic, in each case, bidders may acquire information--or, more generally, incur any costs of participating in the auction--in stages. Specifically, in each case, potential bidders must first decide whether or not to participate in the auction at all. Then, in some cases, bidders must decide how long to participate before dropping out; in other cases, bidders must decide the form--mailed bid versus in person bidding--that their participation will take. These decisions each effect a bidder's total cost of participation.

Other auctions, auctions that we might have tempted to classify as single-stage, also exhibit a similar multi-stage flavor. For example, in an oral auction, a bidder might start with a vague idea of how high he might be willing to bid and only expend the mental energy to refine this limit if and when it looks like he has significant chance of winning the object. Even in a sealed bid auction, a potential bidder might acquire information and refine his estimate of the object's value in stages, stopping this sequential process (and possibly bidding) only when he runs out of time, or when the expected costs of continuing exceed the expected benefits. In fact, all auctions known to this author (with the possible exception of certain, very carefully controlled laboratory experiments) allow for some degree of sequential decision making.

On the other hand, the existing theory of auctions and competitive bidding tends to overlook the sequential nature of bidders' decisions. Indeed, the theory--as surveyed by Engelbrecht-Wiggans (1980) and McAfee and McMillan (1985), or as contributed to by Vickrey (1961), Myerson (1981), and Milgrom and Weber (1982)--focuses on models in which bidders are simply given some specified type and amount of information. Not only don't the bidders acquire information in stages, they have no choice in what or how much information they acquire. (We might call these zero-stage models.) Lee (1984) Matthews (1984), and Engelbrecht-Wiggans (1986) do incorporate a bidder's decision on how much information to buy, or whether to buy any information at all, into their models. However, their models give bidders only a single opportunity to decide how much information they acquire, and might therefore be most appropriately labelled as single-stage models.

In short, the existing theory basically ignores the multi-stage nature of auctions. By not allowing bidders to make sequential decisions on acquiring information--or, more generally, on incurring any costs of participating in the auction--the theory fails to illuminate what or how much information bidders might ultimately acquire at equilibrium. Moreover, the theory cannot illuminate the nature and extent of any benefits to the bidders or to the bid-taker from letting bidders make sequential decisions.

This paper takes a first step in correcting this omission of the current theory. We do so by presenting and solving a specific mathematical example of a multi-stage auction model. In the example, the bid-taker should expect a signficantly greater revenue from bidders at equilibrium (with respect not only to bids, but also with respect to the amount of information acquired) than if all bidders were required to acquire the same amount of information (as would be the case in more traditional models.) More than simply documenting and illustrating one reason why bid-takers might benefit from using multi-stage auctions such as we initially illustrated, this example underscores that sequential decisions can play an important role in an auction and that the multi-stage aspects of real auctions need further study.

Developing an Example:

Several considerations governed our search for a mathematical example. To provide as general and simple an explanation as possible for the existence of multi-stage mechanisms, we 1) assumed the bid-taker and bidders to be risk neutral, and 2) focused on the one common element that we identified for multi-stage auctions--they allow bidders to acquire information in stages. In addition, we only considered cases in which bidders, by acquiring additional information and refining their value estimates, might change who estimates the object to be worth the most; indeed, if bidders would already know by the end of the first stage who would ultimately win the object, then this author would be hard pressed to come up with a simple argument why anyone would ever incur the cost of acquiring additional information. (This means that our example will necessarily differ from the common values models used so far by others when studying the information acquisition process.) Finally, we looked for an example that illuminates the benefits of allowing bidders to acquire their information--and refine their value estimates--in stages, by emphasizing the factors that lead to these benefits, and by suppressing details or considerations that might make the example more "realistic" at the expense of confounding the solution or interpretation of the example.

These considerations led to the following conditions, which, taken together, define the structure of our subsequent example:

1) A single object is offered for sale. Rather than writing "how much more the bidder values the object than the seller values it," we write "the bidder's value" as if the seller had a value of zero for the object. This simplifies the exposition without any loss of generality.

2) To participate in the auction, a bidder must have some private information about his value for the object. Indeed, the work of Engelbrecht-Wiggans, Milgrom and Weber (1983) argues that in a common value setting, a bidder without any private information should expect

zero profit at equilibrium and might as well not bid at all. While this argument applies to many practical situations, it will not apply to our example with its independent information and values; we simply suggest that requiring bidders to have at least some private information, however arbitrary the conditions might seem in our example, has some basis in reality.

3) Individual i has an unknown value v_i equal to $x_i y_i$ for the object. The x_i's are known to be independent outcomes of a random variable X with known cumulative distribution function F(x). Each y_i is known to be equal to unity with a known probability p, and zero otherwise, independent of the x_i's and of the other y_i's.

4) Private information comes in two flavors--imperfect and perfect. By paying a known fee of c_1, individual i sees x_i, giving him an estimate of px_i for his expected value for the object. By paying an additional, known fee of c_2, individual i sees y_i and discovers whether he has a value of zero or x_i for the object. Rather than see x_i and y_i at the same time, at a cost of $c_1 + c_2$, individual i might as well buy the information sequentially. We do not allow anyone to see their y_i without first seeing their x_i.

5) Individuals consider joining the auction (by paying c_1 and seeing their x_i) one by one, knowing how many others have already joined the auction before them. At equilibrium, an individual will pay the fee of c_1 and become a bidder if and only if so doing would give him a non-negative expected profit net of information costs. Thanks to the symmetry of the assumptions, no bidder will ever want to withdraw from the auction simply because others entered the auction after he did and thereby drove down his expected profit.

6) Each bidder submits a single sealed bid. In practice, bidders might submit several bids--the first bid might be based on the information available in the first stage, the second bid on information available in the second stage, and so on. However, in our example, any bidder who, based on his current information, decides to obtain additional information could wait to submit a bid that has any probability of winning until after he has acquired all the information that he ever acquires. Even if we were to require bidders in our example to bid in each stage, a bidder need only bid at most two different amounts--a trivial amount in early stages, and a serious amount once he has all his information. We only care about the single serious amount that each bidder bids. Thus, we assume, without any loss of generality, that each bidder submits only a single bid.

7) If anyone bids at least equal to the known reservation price r, the highest bidder wins the object. Otherwise, the seller keeps the object.

8) The winner, if any, pays the seller an amount equal to the second highest bid, or the reservation price r, whichever is larger. Given the independence of bidders' information, this second price mechanism might be an appropriate model for the commonly used progressive oral auction. More importantly, for this second-price auction with its independent information, no bidder can do better than simply bidding equal to his expected value for the object given whatever information he has acquired; this existence of a dominant strategy bidding equilibrium makes the problem sufficiently tractable so that we can derive the equilibrium information acquisition strategy and fully analyze the example.

Solving the Example:

We consider three cases of the example, and in each case assume X to be distributed uniformly on the unit interval. In the first case, bidders may not acquire perfect information, while in the second case, bidders must acquire perfect information; in fact, these turn out to be single-stage auctions. In the third case, bidders themselves decide whether or not to acquire perfect information; here we will need to derive the equilibrium information acquisition strategy--a strategy which, if followed by all bidders, has each bidder acquire perfect information if and only if so doing increases his expected profit (net of all information costs). We will ultimately see that for at least one specific parameterization of the example, the two-stage mechanism of the third case with a reservation price of zero generates a greater expected revenue for the bid-taker at equilibrium than the single-stage auctions of the first two cases can under any reservation price.

The solution process draws repeatedly on something that we will call the "market value" of the object. In particular, for each fixed number n of bidders, define the market value V(n) as equal to the expected value of the largest of the n bids. Note that the market value implicitly depends on the information acquisition strategy of the bidders; for a fixed information acquisition strategy, the independent private values nature of our example makes V(n) an increasing function of n. We will also refer to the difference between V(n) and the bidders' total costs of information as the "net (social) value"; by the definition of how bidders enter the auction, the bid-taker's expected revenue cannot exceed this net value.

Even with all the simplifying assumptions, solving the example takes considerable effort. We break the solution process into three major steps, described below. We leave the actual derivations to the next section.

1) Calculate the market value $V_I(n)$ for the case in which no one acquires perfect information, and the market value $V_P(n)$ for the case in which all bidders acquire perfect information.

2) For the two-stage case where the bidders decide whether or not to acquire perfect information,

a) Show that for an appropriately chosen constant a, each bidder i following the strategy "acquire perfect information if and only if my x_i at least equals a" yields an equilibrium.

b) For any fixed p, n and a, characterize what c_2 would give rise to this a.

c) Derive the market value $V_T(n)$ for the two-stage auction at equilibrium in terms of p, n, and a (or, equivalently, in terms of p, n, and c_2).

d) Derive the bidders' expected profit at equilibrium and characterize what c_1 would give rise to any specified number n of bidders entering the auction.

The next section contains the actual derivations--these may be skipped by anyone who trusts their accuracy and is interested only in the results. Then in the final section we go to step 3--we consider the case p = a = 1/2 and n = 4. Specifically, we calculate $V_T(n)$ for n = 3 and 4, and use these values to calculate the bid-taker's expected revenue from the two-stage auction with a reserve price of zero and to calculate what c_1 and c_2 would give rise to $a = \frac{1}{2}$ and $n = 4$. For these c_1 and c_2 we then calculate the maximum net value possible in the two single-stage auctions in which all bidders have the same--either perfect or imperfect--level of information. This maximum bounds how much revenue the bid-taker could expect from either single-stage auction. The bound turns out to be significantly less than the bid-taker's expected revenue at equilibrium in the two-stage case.

The Derivations:

This section follows the steps outlined above.

1) If a bidder has only imperfect information, and if each bidder follows the dominant strategy of bidding equal to his expected value for the object given his information, then $V_I(n) = \int_{z=-\infty}^{z=\infty} p \cdot z \cdot \frac{d\Pr(\text{all } x_i\text{'s} \leq z)}{dz} \cdot dz$ $= \frac{pn}{n+1}$ for X distributed uniformly on the unit interval. On the other hand, if each bidder has perfect information, and if each bidder follows the dominant strategy of bidding equal to his expected value for the object given his information, then

$$V_P(n) = E[\max_{0 \leq i \leq n} x_i] = \int_{x=-\infty}^{x=\infty} (1-\Pr(\text{all } x_i\text{'s} \leq z)dz = \int_{z=0}^{z=1} (1-((1-p)+pz)^n))dz$$

$$= 1 - \frac{1}{p(n+1)}(1-(1-p)^{n+1}) \text{ for X distributed uniformly on the unit}$$

interval.

2) For the two-stage case,

a) To verify that "acquire perfect information if and only if my $x_i \geq a$" yields an equilibrium for some a, start by defining $H(y|x) = \Pr(\text{all } n-1 \text{ competing bids} \leq y | x_i = x)$. Note that for our example, $H(y|x)$ is independent of x and may be written as $H(y)$. Now, without perfect information, bidder i would have an expected value $E_I(x_i) = \int_{z \leq px_i} (px_i - z)dH(z)$. On the other hand, if bidder i plans to acquire perfect information, but has not yet done so, he would have an expected value $E_P(x_i) = (1-p)\cdot 0 + p \int_{z \leq x_i} (x_i - z)dH(z)$. But, the expected value of perfect information (over imperfect information) is simply the difference $E_P(x_i) - E_I(x_i)$. Differentiating this difference with respect to x_i yields $p \int_{z \leq x_i} dH(z) - \int_{z \leq px_i} pd(z)$ which simplifies to $p(H(x_i) - H(px_i))$, which must be non-negative since $H(y)$ is a non-decreasing function of y and $0 \leq p \leq 1$. Thus, the expected value of perfect information is a non-decreasing function of x_i, and the strategy "acquire perfect information if and only if my $x_i \geq a$" will be in equilibrium if $c_2 = E_P(x_i = a) - E_I(x_i = a)$.

b) To derive an expression for c_2 in terms of a, p, and n, start by deriving the following explicit expression for H(y):

$$H(y) = \begin{cases} 0 & \text{for } y \leq 0 \\ \left(\frac{y}{p} + (1-a)(1-p)\right)^{n-1} & \text{for } 0 \leq y \leq pa \\ (1-p+pa)^{n-1} & \text{for } pa \leq y \leq a \\ (1-p+py)^{n-1} & \text{for } a \leq y \leq 1 \\ 1 & \text{for } y \geq 1 \end{cases}$$

Now, $c_2 = E_P(a) - E_I(a) = p[\int_{y=0}^{y=pa} (a-y)\frac{d}{dy}(\frac{y}{p} + (1-a)(1-p))^{n-1}dy + 0 + 0] + (1-p)\cdot$

$$- \int_{y=0}^{y=pa} (pa-y)\frac{d}{dy}(\frac{y}{p} + (1-a)(1-p))^{n-1}dy$$

which reduces to $(1-p)\int_{y=0}^{y=pa} y\frac{d}{dy}(\frac{y}{p} + (1-a)(1-p))^{n-1}dy$

and "simplifies" to $\frac{p(1-p)}{n}[(na-1+p-pa)(1-p+pa)^{n-1} + ((1-a)(1-p))^n]$.

For the case of $a = p = 1/2$, $c_2 = \frac{1}{n4^{n+1}}[1+(2n-3)\cdot 3^{n-1}]$.

c) For fixed a, p, and n,

$$V_T(n) = \int_{y=0}^{y=1} y \frac{d}{dy} \Pr(\text{all bids} \leq y)dy = \int_{y=0}^{y=1} (1-\Pr(\text{all bids} \leq y))dy$$

$$= \int_{y=0}^{y=pa} (1-(\frac{y}{p} + (1-a)(1-p))^n)dy + \int_{y=pa}^{y=a} (1-(1-p+pa)^n)dy + \int_{y=a}^{y=1} (1-(1-p+py)^n$$

which "simplifies" to

$$1 - \frac{1}{p(n+1)} + \frac{p((1-a)(1-p))^{n+1}}{n+1} + \frac{(1-p+pa)^n}{n+1} (\frac{1-p+pa}{p} - p(1-p+pa)-a(1-p)(n+1)).$$

For the case of $a = p = 1/2$,

$$V_T(n) = 1 - \frac{2}{n+1} + \frac{1+3^n(7-2n)}{2(n+1)4^{n+1}}.$$

d) In general for second price, independent private values (distributed according to $G(\cdot)$) auctions, the expected market value $V(n)$ equals $\int ynG^{n-1}(y)dG(y)$, and the bid-taker's expected revenue at equilibrium with a reservation price of zero equals $\int yn(n-1)(1-G(y))G(y)^{n-2}dG(y)$, which reduces to $nV(n-1)-(n-1)V(n)$. The difference between these two--namely, $n[V(n)-V(n-1)]$--must equal the bidders' combined expected profit gross of information costs. (Note, as might be expected, each bidder has an expected profit of $V(n)-V(n-1)$--the expected increase in market value due to this additional bidder's presence.) Therefore, to get exactly n participants and to have the last entrant drive each bidder's expected profit (net of information costs) down to exactly zero, we should set $c_1 = V(n)-V(n-1)-(1-G(a))c_2$. (For a slightly smaller c_1, each of the n bidders would have a positive expected profit. As c_1 decreases further, eventually another bidder would enter the market when so doing would again drive the bidders' profits to zero.)

Results and Discussion:

For a = p = 1/2 and X distributed uniformly on the unit interval, $V_T(3) \cong 0.513672$, and $V_T(4) \cong 0.5921875$. This gives the bid-taker an expected revenue of very nearly .2781255 when n = 4. A $c_2 \cong 0.0332031$ gives rise to a = 1/2, while a $c_1 \cong 0.061914$ gives rise to n = 4 bidders participating.

If each bidder has imperfect information, and each must pay c_1 = 0.061914 for his imperfect information, then an n equal to 2 maximizes the net value $V_I(n)-c_1 n$ from this one-stage auction. The maximum net value equals 0.210102. (As it turns out, at a reservation price equal to zero, the 2 bidders' expected profit would be less than their information costs, so the bid-taker would have to subsidize them in order to get two bidders in the auction. For n = 1 and a reservation price of zero, the net value equals 0.198086, and the lone bidder would make a strictly positive expected profit.) Thus, the bid-taker could not have an expected revenue greater than 0.210102 in this one-stage auction, and would probably have to settle for a somewhat smaller expected revenue. Note that the bid-taker's expected revenue in the two-stage auction exceeds this one-stage auction's bound by approximately 32 percent.

Now, if each bidder has perfect information, and each paid $c_1 + c_2$ = 0.0951171 for his information, then the maximum possible net value $V_P(n)-(c_1+(1-a)c_2)$ of 0.2464325 occurs when n = 3. Again, this net value bounds the bid-taker's expected revenue from this one-stage auction. In particular, the expected revenue from the two-stage auction exceeds this second bound by approximately 13 percent--not as much as before, but still a significant amount.

What makes the expected revenue to the bid-taker in the two-stage auction with a reservation price of zero exceed the expected revenue possible at any reservation price in either single-stage auction? Basically, the multi-stage auction allows bidders who are unlikely to win to so discover at a low cost. Thus, for any fixed n, the bidders would spend less on information than if all bidders acquired perfect information. This results in more bidders being able to afford to participate in the auction, thereby driving up the market value above what it would had been had the same amount of money been spent by a smaller number of bidders each acquiring perfect information. As a result, the two-stage auction generates a greater revenue for the bid-taker than the single-stage auction in which all bidders have perfect information.

Alternatively, if all bidders have imperfect information, then there is so large a probability that the bidder who has the highest estimated value is not the bidder who actually has the highest value, that a very inefficient auction results. In fact, even though the imperfect information costs less than the perfect information, the corresponding reduction in auction efficiency and bidders' profits means that in our example the equilibrium in the low information case cannot support as many bidders as the equilibrium when all bidders have perfect information. In total, going from imperfect information to perfect information, increases the market value faster than the bidders' equilibrium costs and profits.

Of course, the bid-taker need not prefer a two-stage auction to a simpler single-stage auction even if it does generate a greater expected revenue. The extra cost of running a two-stage auction may more than offset the gains in expected revenue. Thus, our analysis suggests that bid-takers would be most likely to prefer multi-stage auctions to single-stage auctions when the object to be sold has a large value compared to the costs of running an auction, and the costs of information are large enough so that only a few bidders would tend to participate in the auction. This seems to be the case in the real world illustrations with which we started this paper.

References:

Cassady, R. Jr., Auctions and Auctioneering, University of California Press, Berkeley and Los Angeles, CA, 1967.

Engelbrecht-Wiggans, R., "Auctions and Bidding Models: A Survey," Management Science, Vol. 26, No. 2, February 1980, pp. 119-142.

________, "On Optimal Reservation Prices In Auction," Working Paper, University of Illinois at Urbana-Champaign, November 1986 (to appear in Management Science).

________, P. R. Milgrom and R. J. Weber, "Competitive Bidding and Proprietary Information," Journal of Mathematical Economics, Vol. 11, 1983, pp. 161-169.

Guarino, C. E., "Catalogue #42-95," Americana Arts Auctions, Box 49, Denmark, ME 04022; undated, but issued Fall 1985.

Lee, T. K., "Incomplete Information, High-Low Bidding and Public Information in First Price Auctions," Management Science, Vol. 30, No. 12, pp. 1490-1496, 1984.

Matthews, S. A., "Information Acquisition in Discriminating Auctions," in Bayesian Models in Economic Theory, M. Boyer and R. E. Kihlstrom, eds., Elsevier Science Publishers, 1984.

McAfee, R. P. and J. McMillan, "Auctions and Bidding," Working Paper, University of Western Ontario, December, 1985.

Milgrom, P. R. and R. J. Weber, "A Theory of Auctions and Competitive Bidding," Econometrica, Vol. 50, 1982, pp. 1089-1122.

Myerson, R. B., "Optimal Auction Design," Mathematics of Operations Research, Vol. 6, No. 1, February 1981, pp. 58-73.

Vickrey, W., "Counterspeculation, Auctions and Competitive Sealed Tenders," Journal of Finance, Vol. 41, No. 1, 1961, pp. 8-37.